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The Kernel of the Joint of Two Operators

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Abstract
This study explores the differences between kernel intersections and individual kernel spaces in Hilbert spaces. These differences are crucial for understanding the relationship between linear operators and their compositions. The main findings establish necessary conditions for the inequality dim (Ker (AB)) ≥ dim (Ker(A)) + dim (Ker (B) ∩ B′) to hold, where A and B are bounded linear operators, Ker denotes the kernel, and B′ is the image of B. The proofs rely on Hilbert space properties, closed subspaces, and operator ranges. Although the results are presented in the context of Hilbert spaces, the authors discuss potential extensions to other spaces with similar properties. The paper concludes by emphasizing the broader relevance of these inequalities in various fields of mathematics, including functional analysis, optimization, and potentially other disciplines that deal with continuous quantities.
Keywords: Hilbert spaces, linear operators, kernels, kernel intersections, dimension inequalities, functional analysis, optimization theory

INTRODUCTION

David Hilbert, a renowned German mathematician, made significant contributions to the field of functional analysis, particularly through his development of the notion of a Hilbert space. Born in 1862 and passing away in 1943, Hilbert's work has left an indelible mark on mathematics and its applications, notably in quantum mechanics.

Hilbert's pioneering efforts in functional analysis aimed to extend the concept of Euclidean space to infinite-dimensional spaces. This endeavor culminated in the formulation of what is now known as a Hilbert space. This space serves as a fundamental framework in functional analysis and provides a rich mathematical structure for studying various problems in mathematics and physics.

The theory of Hilbert spaces, developed by Hilbert and other mathematicians, has had far-reaching implications, particularly in the field of quantum mechanics. In quantum mechanics, physical systems are often described using mathematical structures that can be modeled as

Hilbert spaces. This mathematical framework has been instrumental in providing a rigorous foundation for quantum mechanics and has facilitated the development of key concepts and principles in the field.

However, despite the profound impact of Hilbert space theory on quantum mechanics, there remains a gap between the deep mathematical understanding of Hilbert spaces and their application in physics. As noted byLin and Wu (2013), this gap underscores the importance of gaining a thorough mathematical understanding of Hilbert space theory to further advance the development of quantum theory.

One of the defining characteristics of a Hilbert space is its mathematical definition as a complete normed space. This means that a Hilbert space is equipped with a norm (a measure of the size of vectors) that satisfies certain properties, and it is complete in the sense that every Cauchy sequence (a sequence whose elements become arbitrarily close to each other as the sequence progresses) converges to a limit within the space.

Hilbert's mathematical formulations often provided suggestive insights rather than precise problem statements. However, these insights have paved the way for addressing various contemporary mathematical problems Bucur (2003). For instance, Hilbert's work has influenced the development of mathematical subdisciplines such as the theory of quadratic forms and real algebraic curves.

Furthermore, Hilbert's contributions extend to problems concerning the kernel of operations and related mathematical challenges. Understanding and proving properties related to the kernel of operators are essential in various mathematical contexts, including functional analysis and linear algebra.

In summary, David Hilbert's pioneering work in functional analysis, particularly in the development of Hilbert space theory, has had a profound impact on mathematics and physics. His insights have not only enriched the field of mathematics but have also provided invaluable tools for understanding and advancing theories such as quantum mechanics. The rigorous mathematical framework of Hilbert spaces continues to inspire further exploration and development in both theoretical and applied mathematics.

RESEARCH METHODS

- For finding the dim ker (AB) = dim kerB+dim (ker A) "B!)), whether the conditions on A and B which are sufficient to provide dim ker AB>dim ker A.
- Also dim ker AB>dim ker A is valid for all linear operations over finite dimensionalspace through... A, B∈ B (!) <=>dim!< ∞,

Suppose if!be the Hilbert space. Let A,B∈ B (!) be the algebra of hounded linear operators on the Hilbert space H. This gives the conditions on A and B which are sufficient to imply that dim Ker AB ≥ dim KA....................(i)

This conditions are stated as

Statement(1): Suppose B∈ B (!). Then (!) holds for every A∈ B (!) if and only if dimK ≤ dim Ker B for every closed subspace. K⊆! such that K∩ B!= {0} (where K be the closed subspace of!) ....(1.1).

Statement (2): Let B∈ B (!) have closed range. Then (i) holds for every A∈ B (!) if and only if dim (B!) ≤ dim ker B.......... (2.1)

Statement (3): The inequality (i) holds A, B∈ B (!) if and only if d m! < ∞.

Statement (4): A∈ B(!) then dim Ker AB≥ dim Ker A for every B∈ B (!) if and only if (a) Ker A = {0} or (b) dim A! <dim!.

The following example shows that some restriction of the pair A,B is necessary.

Example-1:

Let B ∈ B (!) be one-to-one but not onto. Let K {0} be a closed subspace of! such that K∩B!= {0}. Then (1) fails for every A∈B(!) with kernel K. Although we have not characterized the pairs (A,B) for which (i) holds it is obvious, we have found that the set of B such that (i) holds for every A in B (!) and the A'S such that (i) holds for every B. The utility A (i) may be illustrated by its application in (Erwin), However, there it is stated as though it were true for all A, B∈ B (!). The one aim of this paper is to justify the application if (1) actually made in (Erwin) perhaps the most surprising feature of this note is its involvement with feature of non-closed operator ranges. For this we construct a closed space of maximal dimension which meets the range of a given operator in {0}.

Some useful facts:

Our first fact. Simply lists some routine facts we will need.

Fact (2): Suppose A, Be B (!) and let K, L be closed subspaces of! then dim Ker AB = dim Ker B + dim (Ker A∩B!) ....(2.1)

dim [KY (K∩L] ≤dim L'..............................(2.2)

dim Ker A+ dim A!= dim!............................(2.3)

Fact (3): Let B∈B (!) then ∃closed subspaces K, L such that

K⊂L[1] .... (3.1)

dim K = dim L[1].............. (3.2)

K∩B!= {0} ....(3.3)

and L⊂B (!) ....(3.4)

The main Results:

The Proof of statement (1): Suppose (1) holds for every A∈ B (!). If not, ∃a closed K such that K∩B!= (0) and dim K> dim Ker B. Let A have kernel K. Then dim Ker A> dim Ker B = dim Ker AB.

Conversely: Let K and L be as in the fact (3) then by hypothesis, dim K≤dim Ker B; and so by (2.2) and (3.2) we have dim [KerAŸ (Ker A∩L)] ≤lim L[1]=dim K≤dim Ker B.........(4)

Since Ker A = (Ker A∩L) ⊕[KerAŸ (Ker A∩L)] it follows that dim Ker A≤ dim (kerA∩B !) + dim Ker B = Ker AB by (3.4), (2.1)

Proof of Statement (2)

Setting K = (B!)[1] shows that (1.1) implies that (2.1)

Conversely: Suppose K is closed and K∩B!= {0} since B! is closed it can play the role of L in (2.2). Hence by (2.2) and (2.1) we have dim K dim (B!)'<dim Ker B

Remark: We actually showed that (2.1) is necessary even when B! is not closed

Proof of statement (3)

The example given in (1) shows that dim!<∞ is necessary even if is to hold "A, B∈ B (!) with B! closed.

Conversely -

It dim!< then (2.3) and subtraction (which cannot be justified when dim! = ∞) show that equality holds in (2.1). Next we consider the case where A is fixed.

Proof of statement (4)

Suppose the inequality holds for every B∈B(!) to prove that (a) or (b) must be hold, we shall show that if (b) fails (a) must hold. if (b) fails, dim (Ker A)^1=dimA!=dim!, So there exists an isometry B∈ B (!) of! onto (Ker A)^1 then dim Ker A≤ dim Ker AB = {0}.

This statement will also extendable to the other spaces like as normed space, and their basic features or related problems. The statement plays a central role in the pretty much area of Mathematics or applied Mathematics that deals with continuous quantities, including general analysis, probability, Quantum mechanics, most engineering disciplines and optimization theory. It is not so much used for generating algorithms, but rather for proving that algorithms behave well in some sense. Also for defining the joint kernel from this theory proves the way of higher study of Algebra or real analysis.

Conversely

If (a) holds, the inequality is trivial for every B. Suppose (b) holds and let B∈B(!). Let P be the orthogonal Projection with Kernel Ker A and range (Ker A)^1. Then by (2.3) dim B! = dim Ker ($P|_{BH}$) + dim (PB !)

= dim (Ker A∩B!) + dim (PB!)

<dim (kerA∩B !) + dim (Ker A)^1 .... (5)

If dim Ker B = dim! the validity of the inequality is clear: so assume dim Ker B<dim!. From (2,3) if follows that dim B!= dim! (except in the case that dim! < ∞, where the statement (3) tells us that (1) holds for all A,B∈ B (!). then since dim (Ker A)^1 = dim! <dim!.

We can conclude from (5) that dim (Ker A∩B !) = dim!.then (2.1) completes the proof.

Remark:

Equation (2.1) shows that we there (i) is valid for A, B depends only on Ker A, Ker B, B! and their position!.

RESULT AND DISCUSSION

This paper investigates conditions under which inequalities hold between the dimensions of the intersection of two kernels (Ker(AB)) and the individual kernels (Ker(A) and Ker(B)) in Hilbert spaces. These inequalities offer insights into the relationship between linear operators and their compositions.

The main results establish four separate conditions for the inequality dim(Ker(AB)) ≥ dim(Ker(A)) + dim(Ker(B) ∩ B′) to hold. Here, A and B are bounded linear operators, Ker denotes the kernel, and B′ is the image of B. These conditions focus on properties like closedness of ranges and dimensions of subspaces associated with the operators. The proofs rely on properties of Hilbert spaces and exploit facts about closed subspaces and operator ranges.

While the paper focuses on Hilbert spaces, the authors discuss potential extensions to other spaces with similar characteristics. They highlight the broader applicability of these inequalities in various areas of mathematics, including functional analysis, optimization, and potentially other disciplines dealing with continuous quantities. By providing a deeper understanding of kernel intersections, this work can contribute to advancements in these fields. Future work could involve extending these results to other Banach spaces and investigating the connection between these inequalities and the spectral properties of operators.

CONCLUSION

The question raised by Davis, as highlighted, pertains more naturally to vector spaces and linear transformations rather than being specifically about Hilbert spaces. This observation is crucial because while the statements may suggest a connection to Hilbert space structure, they are applicable to all linear operators over finite-dimensional spaces.

Indeed, even though the statements imply a role for the structure of Hilbert spaces, they hold true in broader contexts. However, it has been convenient for researchers to focus on Hilbert spaces in their analysis. The reason for this convenience lies in the availability of necessary information, particularly regarding non-closed operator ranges, which is more readily accessible in the context of Hilbert spaces.

Filmore and Williams noted this convenience in their work, suggesting that confining the analysis to Hilbert spaces is advantageous due to the clarity of certain basic features, such as non-closed operator ranges, in these spaces. By working within the framework of Hilbert spaces, researchers can leverage the well-established theory and tools available in this setting to address the questions raised by Davis and related problems

Moreover, the expectation is that the theorems developed within the context of Hilbert spaces can be extended to other spaces where the fundamental features, such as non-closed operator ranges, are sufficiently understood. This extension underscores the universality and applicability of the mathematical principles underlying the questions posed by Davis, highlighting the broader relevance of these concepts beyond the specific setting of Hilbert spaces.

In summary, while the questions raised by Davis may seem inherently tied to Hilbert spaces, they have broader implications for vector spaces and linear transformations in general. The choice to analyze these questions within the framework of Hilbert spaces is motivated by the convenience and clarity offered by this setting, particularly concerning essential features like non-closed operator ranges. The expectation is that the results obtained in Hilbert spaces can be extended to other spaces where similar fundamental features are well-understood, thereby demonstrating the universality of the underlying mathematical principles.

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